ARTINIAN AND NON-ARTINIAN LOCAL COHOMOLOGY MODULES

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Abstract. Let M be a finite module over a commutative noetherian ring R. For ideals $\mathfrak{a}$ and $\mathfrak{b}$ of R, the relations between cohomological dimensions of M with respect to $\mathfrak{a}, \mathfrak{b}$, $\mathfrak{a} \cap \mathfrak{b}$ and $\mathfrak{a}+\mathfrak{b}$ are studied. When R is local, it is shown that M is generalized Cohen-Macaulay if there exists an ideal $\mathfrak{a}$ such that all local cohomology modules of M with respect to $\mathfrak{a}$ have finite lengths. Also, when r is an integer such that $0 \leq r < \dim_R(M)$, any maximal element $\mathfrak{q}$ of the non-empty set of ideals $\{\mathfrak{a} : \mathrm{H}^i_{\mathfrak{a}}(M)$ is not artinian for some i, $i \geq r\}$ is a prime ideal and that all Bass numbers of $\mathrm{H}^i_{\mathfrak{q}}(M)$ are finite for all $i \geq r$.

1. Introduction

Throughout R is a commutative noetherian ring, $\mathfrak{a}$ is a proper ideal of R, X and M are non–zero R–modules and M is a finite (i.e. finitely generated). Recall that the ith local cohomology functor $\mathrm{H}^i_{\mathfrak{a}}$ is the ith right derived functor of the $\mathfrak{a}$–torsion functor $\Gamma_{\mathfrak{a}}$. Also, the cohomological dimension of X with respect to $\mathfrak{a}$, denoted by $\mathrm{cd}\,(\mathfrak{a}, X)$, is defined as

$$\mathrm{cd}\,(\mathfrak{a}, X) := \sup\{i : \mathrm{H}^i_{\mathfrak{a}}(X) \neq 0\}.$$

In section 2, we discuss the arithmetic of cohomological dimensions. We show that the inequalities $\mathrm{cd}\,(\mathfrak{a}+\mathfrak{b}, M) \leq \mathrm{cd}\,(\mathfrak{a}, M) + \mathrm{cd}\,(\mathfrak{b}, M)$ and $\mathrm{cd}\,(\mathfrak{a}+\mathfrak{b}, X) \leq \mathrm{ara}(\mathfrak{a}) + \mathrm{cd}\,(\mathfrak{b}, X)$ hold true and we find some equivalent conditions for which each inequality becomes equality.

In section 3, we study artinian local cohomology modules. We first observe that over a local ring $(R, \mathfrak{m})$ if there is an integer n such that $\dim_R(\mathrm{H}^i_{\mathfrak{a}}(X)) \leq 0$ for all $i \leq n$ (respectively, for all $i \geq n$), then $\mathrm{H}^i_{\mathfrak{a}}(X) \cong \mathrm{H}^i_{\mathfrak{m}}(X)$ for all $i \leq n$ (respectively, for all $i \geq n + \mathrm{ara}(\mathfrak{m}/\mathfrak{a})$) (Theorem 3.2). In this situation, if X is finite then $\mathrm{H}^i_{\mathfrak{a}}(X)$ is artinian for all $i \leq n$ (respectively, for all $i \geq n + \mathrm{cd}\,(\mathfrak{m}/\mathfrak{a}, X)$), which is related to the third of Huneke's four problem in local cohomology [11]. Here, for ideals $\mathfrak{a} \subseteq \mathfrak{b}$, $\mathrm{cd}\,(\mathfrak{b}/\mathfrak{a}, X)$ is introduced to be the infimum of the set $\{\mathrm{cd}\,(\mathfrak{c}, X) : \mathfrak{c}$ is an ideal of R and $\sqrt{\mathfrak{b}} = \sqrt{\mathfrak{c}+\mathfrak{a}}\}$. It is deduced that M is generalized Cohen-Macaulay if there exists an ideal $\mathfrak{a}$ such that all local cohomology modules of M with respect to $\mathfrak{a}$ have finite lengths (Corollary 3.4).

Section 4 is devoted to study non-artinian-ness of local cohomology modules. Note that $\mathrm{cd}\,(\mathfrak{a}+Rx, X) \leq \mathrm{cd}\,(\mathfrak{a}, X) + 1$ for all $x \in R$ [9, Lemma 2.5], we show that if there exist $x_1, ..., x_n \in R$ such that $\mathrm{cd}\,(\mathfrak{a}+(x_1, ..., x_n), X) = \mathrm{cd}\,(\mathfrak{a}, X)+n$, then $\dim_R(\mathrm{H}^{\mathrm{cd}\,(\mathfrak{a},X)}_{\mathfrak{a}}(X)) \geq n$ and so $\mathrm{H}^{\mathrm{cd}\,(\mathfrak{a},X)}_{\mathfrak{a}}(X)$ is not artinian (Corollary 4.1). For each integer r, $0 \leq r < d$ ($d :=$

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$\dim_R(M)$), we introduce $\mathcal{L}^r(M)$, the set of all ideals $\mathfrak{a}$ for which $\mathrm{H}^i_{\mathfrak{a}}(M)$ is not artinian for some $i \geq r$. It is evident that if $d > 0$ then $\mathcal{L}^r(M)$ is not empty. We show that any maximal element $\mathfrak{q}$ of $\mathcal{L}^r(M)$ is a prime ideal and that all Bass numbers of $\mathrm{H}^i_{\mathfrak{q}}(M)$ are finite for all $i \geq r$. We conclude that this statement generalizes [5, Corollary 2] (see Theorem 4.7 and its comment).

2. Arithmetic of cohomological dimensions

Assume that $\mathfrak{a}, \mathfrak{b}$ are ideals of R and that X is an R–module. In this section, we study relationships between the numbers $\mathrm{cd}\,(\mathfrak{a}, X), \mathrm{cd}\,(\mathfrak{b}, X), \mathrm{cd}\,(\mathfrak{a}+\mathfrak{b}, X), \mathrm{cd}\,(\mathfrak{a}\cap\mathfrak{b}, X)(= \mathrm{cd}\,(\mathfrak{a}\mathfrak{b}, X))$, $\mathrm{ara}(\mathfrak{a})$, etc, which are interesting in themselves and we use them to determine artinian-ness and non-artinian-ness of certain local cohomology modules in the next sections.

Lemma 2.1. *Let X be an R–module and let t be a non-negative integer such that for all r, $0 \leq r \leq t$, $\mathrm{H}^{t-r}_{\mathfrak{a}}(\mathrm{H}^r_{\mathfrak{b}}(X)) = 0$. Then $\mathrm{H}^t_{\mathfrak{a}+\mathfrak{b}}(X)$ is also zero.*

Proof. By [14, Theorem 11.38], there is a Grothendieck spectral sequence

$$E_2^{p,q} := \mathrm{H}^p_{\mathfrak{a}}(\mathrm{H}^q_{\mathfrak{b}}(X)) \underset{p}{\Longrightarrow} \mathrm{H}^{p+q}_{\mathfrak{a}+\mathfrak{b}}(X).$$

For all r, $0 \leq r \leq t$, we have $E_\infty^{t-r,r} = E_{t+2}^{t-r,r}$ since $E_i^{t-r-i,r+i-1} = 0 = E_i^{t-r+i,r+1-i}$ for all $i \geq t+2$. Note that $E_{t+2}^{t-r,r}$ is a subquotient of $E_2^{t-r,r}$ which is zero by assumption. Thus $E_{t+2}^{t-r,r}$ is zero, that is $E_\infty^{t-r,r} = 0$. There exists a finite filtration

$$0 = \phi^{t+1}H^t \subseteq \phi^t H^t \subseteq \cdots \subseteq \phi^1 H^t \subseteq \phi^0 H^t = \mathrm{H}^t_{\mathfrak{a}+\mathfrak{b}}(X)$$

such that $E_\infty^{t-r,r} = \phi^{t-r}H^t/\phi^{t-r+1}H^t$ for all r, $0 \leq r \leq t$. Therefore we have

$$0 = \phi^{t+1}H^t = \phi^t H^t = \cdots = \phi^1 H^t = \phi^0 H^t = \mathrm{H}^t_{\mathfrak{a}+\mathfrak{b}}(X)$$

as desired. □

The following corollary is the first application of the above lemma.

Corollary 2.2. *For a finite R–module M, the following statements hold true.*

(i) $\mathrm{cd}\,(\mathfrak{a}+\mathfrak{b}, M) \leq \mathrm{cd}\,(\mathfrak{a}, M) + \mathrm{cd}\,(\mathfrak{b}, M)$.

(ii) $\mathrm{cd}\,(\mathfrak{a}\cap\mathfrak{b}, M) \leq \mathrm{cd}\,(\mathfrak{a}, M) + \mathrm{cd}\,(\mathfrak{b}, M)$.

(iii) $\mathrm{cd}\,(\mathfrak{a}, M) \leq \sum_{\mathfrak{p}\in\mathrm{Min}\,(\mathfrak{a})} \mathrm{cd}\,(\mathfrak{p}, M)$.

Proof. (i) Assume that t is a non-negative integer such that $t > \mathrm{cd}\,(\mathfrak{a}, M) + \mathrm{cd}\,(\mathfrak{b}, M)$. We show that $\mathrm{H}^{t-r}_{\mathfrak{a}}(\mathrm{H}^r_{\mathfrak{b}}(M)) = 0$ for all r, $0 \leq r \leq t$. If $r > \mathrm{cd}\,(\mathfrak{b}, M)$, then $\mathrm{H}^{t-r}_{\mathfrak{a}}(\mathrm{H}^r_{\mathfrak{b}}(M)) = 0$ by the definition of cohomological dimension. Otherwise, $t - r > \mathrm{cd}\,(\mathfrak{a}, M)$. Since $\mathrm{Supp}_R(\mathrm{H}^r_{\mathfrak{b}}(M)) \subseteq \mathrm{Supp}_R(M)$, $\mathrm{cd}\,(\mathfrak{a}, M) \geqslant \mathrm{cd}\,(\mathfrak{a}, \mathrm{H}^r_{\mathfrak{b}}(M))$ (see [6, Theorem 1.4]). Therefore $\mathrm{H}^{t-r}_{\mathfrak{a}}(\mathrm{H}^r_{\mathfrak{b}}(M)) = 0$. Now applying Lemma 2.1, we see that $\mathrm{H}^t_{\mathfrak{a}+\mathfrak{b}}(M) = 0$ which yields the assertion.

(ii) Consider the Mayer-Vietoris exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}+\mathfrak{b}}(M) \longrightarrow \Gamma_{\mathfrak{a}}(M) \oplus \Gamma_{\mathfrak{b}}(M) \longrightarrow \Gamma_{\mathfrak{a}\cap\mathfrak{b}}(M) \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathrm{H}^t_{\mathfrak{a}+\mathfrak{b}}(M) \longrightarrow \mathrm{H}^t_{\mathfrak{a}}(M) \oplus \mathrm{H}^t_{\mathfrak{b}}(M) \longrightarrow \mathrm{H}^t_{\mathfrak{a}\cap\mathfrak{b}}(M) \longrightarrow \mathrm{H}^{t+1}_{\mathfrak{a}+\mathfrak{b}}(M) \longrightarrow \cdots$$

and use part (i).

(iii) As $\sqrt{\mathfrak{a}} = \bigcap\limits_{\mathfrak{p}\in \mathrm{Min}\,(\mathfrak{a})} \mathfrak{p}$, the claim follows from part (ii). □

Remark 2.3. In the above corollary, one may state more precise statements is certain cases as follows:

(ii′) If $\mathrm{cd}\,(\mathfrak{a}, M) > 0$ and $\mathrm{cd}\,(\mathfrak{b}, M) > 0$, then $\mathrm{cd}\,(\mathfrak{a}\cap\mathfrak{b}, M) \leq \mathrm{cd}\,(\mathfrak{a}, M) + \mathrm{cd}\,(\mathfrak{b}, M) - 1$.

(iii′) If R is local and M is not $\mathfrak{a}$–torsion, then

$$\mathrm{cd}\,(\mathfrak{a}, M) \leq \sum_{\mathfrak{p}\in \mathrm{Min}\,(\mathfrak{a})} \mathrm{cd}\,(\mathfrak{p}, M) - |\mathrm{Min}\,(\mathfrak{a})| + 1.$$

Note that the proof of (ii′) is similar to that of Corollary 2.2(ii). For (iii′), we have $\mathrm{cd}\,(\mathfrak{p}, M) > 0$ for all $\mathfrak{p} \in \mathrm{Min}\,(\mathfrak{a})$ since M is not $\mathfrak{a}$–torsion. The result follows by induction on $|\mathrm{Min}\,(\mathfrak{a})|$.

For a general module X, not necessarily finite, we have the following result.

Corollary 2.4. *Let X be an arbitrary R–module. Then the following statements hold true.*

(i) $\mathrm{cd}\,(\mathfrak{a}+\mathfrak{b}, X) \leq \mathrm{ara}(\mathfrak{a}) + \mathrm{cd}\,(\mathfrak{b}, X)$.

(ii) $\mathrm{cd}\,(\mathfrak{a}\cap\mathfrak{b}, X) \leq \mathrm{ara}(\mathfrak{a}) + \mathrm{cd}\,(\mathfrak{b}, X)$.

(iii) $\mathrm{cd}\,(\mathfrak{b}, X) \leq \mathrm{cd}\,(\mathfrak{a}, X) + \mathrm{ara}(\mathfrak{b}/\mathfrak{a})$ *whenever* $\mathfrak{a} \subseteq \mathfrak{b}$.

Proof. The proofs of (i) and (ii) are similar to those of Corollary 2.2 (i) and (ii), respectively. For (iii), let $e = \mathrm{cd}\,(\mathfrak{a}, X)$ and $f = \mathrm{ara}(\mathfrak{b}/\mathfrak{a})$. There exist $x_1, ..., x_f \in R$ such that $\sqrt{\mathfrak{b}} = \sqrt{(x_1, ..., x_f) + \mathfrak{a}}$. Now, use part (i). □

We need some sufficient conditions for validity of the isomorphism $\mathrm{H}^s_{\mathfrak{a}}(\mathrm{H}^t_{\mathfrak{b}}(X)) \cong \mathrm{H}^{s+t}_{\mathfrak{a}+\mathfrak{b}}(X)$, for given non-negative integers s and t, which is crucial for the rest of the paper, e.g. to determine equalities in Corollary 2.2(i) and Corollary 2.4(i).

Lemma 2.5. *Let X be an arbitrary R–module and let s, t be non-negative integers such that*

(a) $\mathrm{H}^{s+t-i}_{\mathfrak{a}}(\mathrm{H}^i_{\mathfrak{b}}(X)) = 0$ *for all* $i \in \{0, \cdots, s+t\} \setminus \{t\}$,

(b) $\mathrm{H}^{s+t-i+1}_{\mathfrak{a}}(\mathrm{H}^i_{\mathfrak{b}}(X)) = 0$ *for all* $i \in \{0, \cdots, t-1\}$, *and*

(c) $\mathrm{H}^{s+t-i-1}_{\mathfrak{a}}(\mathrm{H}^i_{\mathfrak{b}}(X)) = 0$ *for all* $i \in \{t+1, \cdots, s+t\}$.

Then we have the isomorphism $\mathrm{H}^s_{\mathfrak{a}}(\mathrm{H}^t_{\mathfrak{b}}(X)) \cong \mathrm{H}^{s+t}_{\mathfrak{a}+\mathfrak{b}}(X)$.

Proof. Consider the Grothendieck spectral sequence

$$E_2^{p,q} := \mathrm{H}^p_{\mathfrak{a}}(\mathrm{H}^q_{\mathfrak{b}}(X)) \underset{p}{\Longrightarrow} \mathrm{H}^{p+q}_{\mathfrak{a}+\mathfrak{b}}(X).$$

For all $r \geq 2$, let $Z_r^{s,t} = \ker(E_r^{s,t} \longrightarrow E_r^{s+r,t+1-r})$ and $B_r^{s,t} = \mathrm{Im}\,(E_r^{s-r,t+r-1} \longrightarrow E_r^{s,t})$. We have exact sequences

$$0 \longrightarrow B_r^{s,t} \longrightarrow Z_r^{s,t} \longrightarrow E_{r+1}^{s,t} \longrightarrow 0$$

and

$$0 \longrightarrow Z_r^{s,t} \longrightarrow E_r^{s,t} \longrightarrow E_r^{s,t}/Z_r^{s,t} \longrightarrow 0.$$

Since, by assumptions (b) and (c), $E_2^{s+r,t+1-r} = 0 = E_2^{s-r,t+r-1}$, $E_r^{s+r,t+1-r} = 0 = E_r^{s-r,t+r-1}$. Therefore $E_r^{s,t}/Z_r^{s,t} = 0 = B_r^{s,t}$ which shows that $E_r^{s,t} = E_{r+1}^{s,t}$ and so

$$\mathrm{H}_{\mathfrak{a}}^s(\mathrm{H}_{\mathfrak{b}}^t(X)) = E_2^{s,t} = E_3^{s,t} = \cdots = E_{s+t+1}^{s,t} = E_{s+t+2}^{s,t} = E_\infty^{s,t}.$$

There is a finite filtration

$$0 = \phi^{s+t+1}H^{s+t} \subseteq \phi^{s+t}H^{s+t} \subseteq \cdots \subseteq \phi^1 H^{s+t} \subseteq \phi^0 H^{s+t} = H_{\mathfrak{a}+\mathfrak{b}}^{s+t}(X)$$

such that $E_\infty^{s+t-r,r} = \phi^{s+t-r}H^{s+t}/\phi^{s+t-r+1}H^{s+t}$ for all r, $0 \le r \le s+t$.

Note that for each r, $0 \le r \le t-1$ or $t+1 \le r \le s+t$, $E_\infty^{s+t-r,r} = 0$ by assumption (a). Therefore we get

$$0 = \phi^{s+t+1}H^{s+t} = \phi^{s+t}H^{s+t} = \cdots = \phi^{s+2}H^{s+t} = \phi^{s+1}H^{s+t}$$

and

$$\phi^s H^{s+t} = \phi^{s-1}H^{s+t} = \cdots = \phi^1 H^{s+t} = \phi^0 H^{s+t} = \mathrm{H}_{\mathfrak{a}+\mathfrak{b}}^{s+t}(X).$$

Hence $\mathrm{H}_{\mathfrak{a}}^s(\mathrm{H}_{\mathfrak{b}}^t(X)) = E_\infty^{s,t} = \phi^s H^{s+t}/\phi^{s+1}H^{s+t} = \mathrm{H}_{\mathfrak{a}+\mathfrak{b}}^{s+t}(X)$ as desired. □

Now, we are able to discuss conditions under which the inequalities Corollary 2.2(i) and Corollary 2.4(i) become equalities.

Corollary 2.6. *Suppose that M is a finite R–module such that $(\mathfrak{a}+\mathfrak{b})M \neq M$. Then the following statements hold true.*

(i) $\mathrm{H}_{\mathfrak{a}+\mathfrak{b}}^{\mathrm{cd}\,(\mathfrak{a},M)+\mathrm{cd}\,(\mathfrak{b},M)}(M) \cong \mathrm{H}_{\mathfrak{a}}^{\mathrm{cd}\,(\mathfrak{a},M)}(\mathrm{H}_{\mathfrak{b}}^{\mathrm{cd}\,(\mathfrak{b},M)}(M))$.

(ii) *The following statements are equivalent.*

 (a) $\mathrm{cd}\,(\mathfrak{a}+\mathfrak{b},M) = \mathrm{cd}\,(\mathfrak{a},M) + \mathrm{cd}\,(\mathfrak{b},M)$.

 (b) $\mathrm{cd}\,(\mathfrak{a},M) = \mathrm{cd}\,(\mathfrak{a},\mathrm{H}_{\mathfrak{b}}^{\mathrm{cd}\,(\mathfrak{b},M)}(M))$.

 (c) $\mathrm{cd}\,(\mathfrak{b},M) = \mathrm{cd}\,(\mathfrak{b},\mathrm{H}_{\mathfrak{a}}^{\mathrm{cd}\,(\mathfrak{a},M)}(M))$.

Proof. (i) Apply Lemma 2.5 with $s = \mathrm{cd}\,(\mathfrak{a},M)$ and $t = \mathrm{cd}\,(\mathfrak{b},M)$.

(ii) The implications $(a \Rightarrow b)$ and $(a \Rightarrow c)$ are clear from part (i) and [6, Theorem 1.4]. For implications $(b \Rightarrow a)$ and $(c \Rightarrow a)$, one may use part (i) and Corollary 2.2(i). □

With a similar argument, one has the following result for an arbitrary module.

Corollary 2.7. *Suppose that X is an arbitrary R–module. Then we have*

(i) $\mathrm{H}_{\mathfrak{a}+\mathfrak{b}}^{\mathrm{ara}(\mathfrak{a})+\mathrm{cd}\,(\mathfrak{b},X)}(X) \cong \mathrm{H}_{\mathfrak{a}}^{\mathrm{ara}(\mathfrak{a})}(\mathrm{H}_{\mathfrak{b}}^{\mathrm{cd}\,(\mathfrak{b},X)}(X))$.

(ii) *The following statements are equivalent.*

 (a) $\mathrm{cd}\,(\mathfrak{a}+\mathfrak{b},X) = \mathrm{ara}(\mathfrak{a}) + \mathrm{cd}\,(\mathfrak{b},X)$.

 (b) $\mathrm{ara}(\mathfrak{a}) = \mathrm{cd}\,(\mathfrak{a},\mathrm{H}_{\mathfrak{b}}^{\mathrm{cd}\,(\mathfrak{b},X)}(X))$.

3. ARTINIAN LOCAL COHOMOLOGY MODULES

In this section, we study artinian property of local cohomology modules. For this purpose, for ideals $\mathfrak{b} \supseteq \mathfrak{a}$, we introduce the notion of cohomological dimension of an R–module X with respect to $\mathfrak{b}/\mathfrak{a}$.

Definition 3.1. *Let $\mathfrak{b} \supseteq \mathfrak{a}$ be ideals of R and let X be an R–module. Define the cohomological dimension of X with respect to $\mathfrak{b}/\mathfrak{a}$ as*

$$\operatorname{cd}(\mathfrak{b}/\mathfrak{a}, X) := \inf\{\operatorname{cd}(\mathfrak{c}, X) : \mathfrak{c} \text{ is an ideal of } R \text{ and } \sqrt{\mathfrak{b}} = \sqrt{\mathfrak{c}+\mathfrak{a}}\}.$$

It is easy to see that $\operatorname{cd}(\mathfrak{b}/\mathfrak{a}, X) \leq \operatorname{ara}(\mathfrak{b}/\mathfrak{a})$ and, for a finite R–module M,

$$\operatorname{cd}(\mathfrak{b}/\mathfrak{a}, M) \geq \operatorname{cd}(\mathfrak{b}, M) - \operatorname{cd}(\mathfrak{a}, M)$$

by Corollary 2.2(i). Note that when $\mathfrak{a}X = 0$, we have $\operatorname{cd}(\mathfrak{b}/\mathfrak{a}, X) = \operatorname{cd}(\mathfrak{b}, X) = \operatorname{cd}_{R/\mathfrak{a}}(\mathfrak{b}/\mathfrak{a}, X)$. One may notice that if $\operatorname{Supp}_R(X) \subseteq \operatorname{Supp}_R(M)$, then $\operatorname{cd}(\mathfrak{b}/\mathfrak{a}, X) \leq \operatorname{cd}(\mathfrak{b}/\mathfrak{a}, M)$.

Now, we can state the following theorem.

Theorem 3.2. *Let $\mathfrak{b} \supseteq \mathfrak{a}$ be ideals of R, let X be an arbitrary R–module and let n be a non-negative integer.*

(i) *If $\mathrm{H}^i_{\mathfrak{a}}(X)$ is $\mathfrak{b}$–torsion for all i, $0 \leq i \leq n$, then $\mathrm{H}^i_{\mathfrak{a}}(X) \cong \mathrm{H}^i_{\mathfrak{b}}(X)$ for all i, $0 \leq i \leq n$.*
(ii) *If $\mathrm{H}^i_{\mathfrak{a}}(X)$ is $\mathfrak{b}$–torsion for all $i \geq n$, then $\mathrm{H}^i_{\mathfrak{a}}(X) \cong H^i_{\mathfrak{b}}(X)$ for all $i \geq n + \operatorname{ara}(\mathfrak{b}/\mathfrak{a})$.*
(iii) *Assume that M is a finite R–module and that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{b}$–torsion for all $i \geq n$. Then $\mathrm{H}^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{b}}(M)$ for all $i \geq n + \operatorname{cd}(\mathfrak{b}/\mathfrak{a}, M)$.*

Proof. Let $u = \operatorname{ara}(\mathfrak{b}/\mathfrak{a})$ and $v = \operatorname{cd}(\mathfrak{b}/\mathfrak{a}, M)$. There exist $x_1, ..., x_u \in R$ and an ideal $\mathfrak{c}$ of R such that $\operatorname{cd}(\mathfrak{c}, M) = v$ and $\sqrt{(x_1, ..., x_u) + \mathfrak{a}} = \sqrt{\mathfrak{b}} = \sqrt{\mathfrak{c}+\mathfrak{a}}$. In computing local cohomology modules, we may assume that $(x_1, ..., x_u) + \mathfrak{a} = \mathfrak{b} = \mathfrak{c} + \mathfrak{a}$. Now, for all i, $0 \leq i \leq n$, (respectively, $i \geq n + u$, $i \geq n + v$,) apply Lemma 2.5 with $s = 0$ and $t = i$ to obtain the isomorphisms $\Gamma_{(x_1,...,x_u)}(\mathrm{H}^i_{\mathfrak{a}}(X)) \cong \mathrm{H}^i_{\mathfrak{b}}(X)$ for all i, $0 \leq i \leq n$, (respectively, $\Gamma_{(x_1,...,x_u)}(\mathrm{H}^i_{\mathfrak{a}}(X)) \cong \mathrm{H}^i_{\mathfrak{b}}(X)$ for all $i \geq n + u$, $\Gamma_{\mathfrak{c}}(\mathrm{H}^i_{\mathfrak{a}}(M)) \cong \mathrm{H}^i_{\mathfrak{b}}(M)$ for all $i \geq n + v$,). Therefore all of the assertions follow. □

Corollary 3.3. *Let R be a local ring with maximal ideal $\mathfrak{m}$, let M be a finite R–module and let n be a non-negative integer. If $\dim_R(H^i_{\mathfrak{a}}(M)) \leq 0$ for all i, $0 \leq i \leq n$ (respectively, for all $i \geq n$), then $H^i_{\mathfrak{a}}(M)$ is artinian for all i, $0 \leq i \leq n$ (respectively, for all $i \geq n + \operatorname{cd}(\mathfrak{m}/\mathfrak{a}, M)$).*

Proof. Since $\mathrm{H}^i_{\mathfrak{m}}(M)$ is artinian for all i, the assertion follows from Theorem 3.2. □

Recall that a finite R–module M over a local ring $(R, \mathfrak{m})$ is called a *generalized Cohen–Macaulay* module if $\mathrm{H}^i_{\mathfrak{m}}(M)$ is of finite length for all $i < \dim_R(M)$. The following result gives us a characterization for a finite module M over a local ring to be generalized Cohen-Macaulay in terms of the existence of an ideal $\mathfrak{a}$ for which $\mathrm{H}^i_{\mathfrak{a}}(M)$ is of finite length for all $i < \dim_R(M)$.

Corollary 3.4. *Let R be a local ring with maximal ideal $\mathfrak{m}$ and let M be a finite R–module. Then the following statements are equivalent.*

(i) *M is generalized Cohen-Macaulay module.*
(ii) *There exists an ideal $\mathfrak{a}$ such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is of finite length for all i, $0 \leq i < \dim_R(M)$.*

Proof. (i) $\Rightarrow$ (ii). It is trivial.

(ii) $\Rightarrow$ (i). This follows from Theorem 3.2(i). $\square$

A non-zero R–module X is called *secondary* if its multiplication map by any element a of R is either surjective or nilpotent. A prime ideal $\mathfrak{p}$ of R is said to be an *attached prime* of X if $\mathfrak{p} = (T :_R X)$ for some submodule T of X. If X admits a reduced secondary representation, $X = X_1 + X_2 + \cdots + X_n$, then the set of attached primes $\mathrm{Att}\,_R(X)$ of X is equal to $\{\sqrt{0 :_R X_i} : i = 1, \cdots, n\}$ (cf. [12]).

Assume that M is a finite R–module of finite dimension d and that $\mathfrak{a}$ is an ideal of R. It is well-known that $\mathrm{H}^d_{\mathfrak{a}}(M)$ is artinian. If $(R, \mathfrak{m})$ is local, then the first author and Yassemi in [7, Theorem A] (see also [10, Theorem 8.2.1]) showed that $\mathrm{Att}\,_R(\mathrm{H}^d_{\mathfrak{a}}(M)) = \{\mathfrak{p} \in \mathrm{Assh}\,_R(M) : \mathrm{H}^d_{\mathfrak{a}}(R/\mathfrak{p}) \neq 0\}$ which generalized the well-known result $\mathrm{Att}\,_R(\mathrm{H}^d_{\mathfrak{m}}(M)) = \mathrm{Assh}\,_R(M)(= \{\mathfrak{p} \in \mathrm{Supp}\,_R(M) : \dim(R/\mathfrak{p}) = d\})$ (see [13, Theorem 2.2]). In the following remark, those ideals $\mathfrak{a}$ for which $\mathrm{Att}\,_R(\mathrm{H}^d_{\mathfrak{a}}(M)) = \mathrm{Assh}\,_R(M)$ are characterized. Denote the height support, $\mathrm{hSupp}\,_R(M)$, of M as the set of all $\mathfrak{p} \in \mathrm{Supp}\,_R(M)$ such that $\mathfrak{p} \in \mathrm{V}(\mathfrak{q})$ for some $\mathfrak{q} \in \mathrm{Assh}\,_R(M)$.

Remark 3.5. Let $(R, \mathfrak{m})$ be a complete local ring and let M be a non-zero finite R–module with Krull dimension d. Then the following statements are equivalent.

(i) $\mathrm{H}^d_{\mathfrak{a}}(M) \cong \mathrm{H}^d_{\mathfrak{m}}(M)$.

(ii) $\mathrm{Att}\,_R(\mathrm{H}^d_{\mathfrak{a}}(M)) = \mathrm{Assh}\,_R(M)$.

(iii) $\mathrm{V}(\mathfrak{a}) \cap \mathrm{hSupp}\,_R(M) = \{\mathfrak{m}\}$.

The proof of (i) $\Rightarrow$ (ii) is clear. To prove (ii) $\Rightarrow$ (iii), one may use Lichtenbaum-Hartshorne Vanishing Theorem. For (iii) $\Rightarrow$ (i), choose a submodule N of M such that $\mathrm{Ass}\,_R(N) = \mathrm{Ass}\,_R(M) \setminus \mathrm{Assh}\,_R(M)$ and $\mathrm{Ass}\,_R(M/N) = \mathrm{Assh}\,_R(M)$ to obtain $\mathrm{H}^d_{\mathfrak{a}}(M) \cong \mathrm{H}^d_{\mathfrak{a}}(M/N)$ and $\mathrm{H}^d_{\mathfrak{m}}(M) \cong \mathrm{H}^d_{\mathfrak{m}}(M/N)$. Therefore $\mathrm{Supp}\,_R(\mathrm{H}^i_{\mathfrak{a}}(M/N)) \subseteq \{\mathfrak{m}\}$ for all i. Applying Theorem 3.2(i) gives the claim. This remark shows that if M is equidimensional, then $\mathrm{Att}\,_R(\mathrm{H}^d_{\mathfrak{a}}(M)) \neq \mathrm{Assh}\,_R(M)$ for each ideal $\mathfrak{a}$ with $\mathrm{ht}\,_M(\mathfrak{a}) < \dim_R(M)$.

Recall that, an R–module X is said to be *minimax* if it has a finite submodule X' such that X/X' is artinian (See [15]). Note that the class of minimax modules includes all finite and all artinian modules. We close this section by showing that if $\mathfrak{m}$ is a maximal ideal containing $\mathfrak{a}$, then $\mathrm{H}^i_{\mathfrak{m}}(M)$ is artinian for all $i \leq n$ (respectively, for all $i \geq n + \mathrm{cd}\,(\mathfrak{m}/\mathfrak{a}, M)$) whenever $\mathrm{H}^i_{\mathfrak{a}}(M)$ is minimax for all $i \leq n$ (respectively, for all $i \geq n$). We first bring an analogous lemma as Lemma 2.1.

Lemma 3.6. *Let X be an R–module and let t be a non-negative integer such that $\mathrm{H}^{t-r}_{\mathfrak{a}}(\mathrm{H}^r_{\mathfrak{b}}(X))$ is artinian for all r, $0 \leq r \leq t$. Then $\mathrm{H}^t_{\mathfrak{a}+\mathfrak{b}}(X)$ is artinian.*

Proof. By the Grothendieck spectral sequence

$$E_2^{p,q} := \mathrm{H}^p_{\mathfrak{a}}(\mathrm{H}^q_{\mathfrak{b}}(X)) \underset{p}{\Rightarrow} \mathrm{H}^{p+q}_{\mathfrak{a}+\mathfrak{b}}(X),$$

the proof is similar to that of Lemma 2.1. $\square$

Theorem 3.7. *Let $\mathfrak{m}$ be a maximal ideal of R contains $\mathfrak{a}$, let X be an arbitrary R–module and let n be a non-negative integer. Then*

(i) *If $\mathrm{H}^i_{\mathfrak{a}}(X)$ is minimax for all i, $0 \leq i \leq n$, then $\mathrm{H}^i_{\mathfrak{m}}(X)$ is artinian for all i, $0 \leq i \leq n$.*
(ii) *If $\mathrm{H}^i_{\mathfrak{a}}(X)$ is minimax for all $i \geq n$, then $\mathrm{H}^i_{\mathfrak{m}}(X)$ is artinian for all $i \geq n + \mathrm{ara}(\mathfrak{m}/\mathfrak{a})$.*
(iii) *Assume that M is a finite R–module and that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is minimax for all $i \geq n$. Then $\mathrm{H}^i_{\mathfrak{m}}(M)$ is artinian for all $i \geq n + \mathrm{cd}\,(\mathfrak{m}/\mathfrak{a}, M)$.*

Proof. By considering lemma 3.6, this is similar to that of Theorem 3.2. □

4. Non-artinian local cohomology modules

In this section, we study those local cohomology modules which are not artinian. The following two results give us many non-artinian local cohomology modules.

Corollary 4.1. *Let X be an R–module, let n be a positive integer and let $x_1, ..., x_n \in R$ such that $\mathrm{cd}\,(\mathfrak{a} + (x_1, ..., x_n), X) = \mathrm{cd}\,(\mathfrak{a}, X) + n$. Then $\dim_R(\mathrm{H}^{\mathrm{cd}\,(\mathfrak{a},X)}_{\mathfrak{a}}(X)) \geq n$. In particular, $\mathrm{H}^{\mathrm{cd}\,(\mathfrak{a},X)}_{\mathfrak{a}}(X)$ is not artinian.*

Proof. By Corollary 2.4(i), $\mathrm{ara}(x_1, ..., x_n) = n$. By Corollary 2.7(ii) and Grothendieck Vanishing Theorem, we have $\dim_R(\mathrm{H}^{\mathrm{cd}\,(\mathfrak{a},X)}_{\mathfrak{a}}(X)) \geq n$ and so $\mathrm{H}^{\mathrm{cd}\,(\mathfrak{a},X)}_{\mathfrak{a}}(X)$ is not artinian. □

Corollary 4.2. (cf. [2, Proposition 3.2]) *Let $(R, \mathfrak{m})$ be a local ring and let M be a finite R–module with Krull dimension d. Assume also that $\mathfrak{a}$ is generated by a subset of system of parameters $x_1, ..., x_n$ of M of length n. Then $\dim_R(\mathrm{H}^{\mathrm{cd}\,(\mathfrak{a},M)}_{\mathfrak{a}}(M)) = d - n$. In particular, if $n < d$, then $\mathrm{H}^{\mathrm{cd}\,(\mathfrak{a},M)}_{\mathfrak{a}}(M)$ is not artinian.*

Proof. There exist $x_{n+1}, ..., x_d \in R$ such that $x_1, ..., x_d$ is a system of parameters of M. Set $\mathfrak{b} = (x_{n+1}, ..., x_d)$. As $\mathfrak{m} = \sqrt{\mathfrak{a} + \mathfrak{b} + \mathrm{Ann}_R(M)}$, we can, and do, assume that $\mathfrak{a} + \mathfrak{b} = \mathfrak{m}$. By Corollary 2.2(i), $\mathrm{cd}\,(\mathfrak{a}, M) = n$ and $\mathrm{cd}\,(\mathfrak{b}, M) = d - n$. Now, by using Corollary 2.6(ii), we obtain $\dim_R(\mathrm{H}^n_{\mathfrak{a}}(M)) \geqslant d - n$. On the other hand, we have $\dim_R(\mathrm{H}^n_{\mathfrak{a}}(M)) \leqslant d - n$ since $\mathrm{Supp}_R(\mathrm{H}^n_{\mathfrak{a}}(M)) \subseteq \mathrm{Supp}_R(M/\mathfrak{a}M)$. Thus $\dim_R(\mathrm{H}^n_{\mathfrak{a}}(M)) = d - n$ as desired. □

Now it is natural to raise the following question.

Question 4.3. *Assume that M is a finite R–module and that $\mathrm{H}^{\mathrm{cd}\,(\mathfrak{a},M)}_{\mathfrak{a}}(M)$ is not artinian. Is there an element x in R such that*

$$\mathrm{cd}\,(\mathfrak{a} + Rx, M) = \mathrm{cd}\,(\mathfrak{a}, M) + 1?$$

It is clear that the above question has a positive answer if R is local and $\mathfrak{a}$ is generated by a subset of system of parameters of M of length smaller than $\dim_R(M)$.

In the rest of the paper, we study the set of ideals $\mathfrak{b}$ of R such that $\mathrm{H}^i_{\mathfrak{b}}(M)$ is not artinian for some non-negative integer i.

Definition 4.4. *Let M be a finite R–module and let r be a non-negative integer. Define the set of ideals*

$$\mathcal{L}^r(M) := \{\mathfrak{b} : H^i_{\mathfrak{b}}(M) \text{ is not artinian for some } i \geq r\}.$$

Note that $\mathcal{L}^r(M)$ is the empty set for all $r \geq \dim_R(M)$. If $0 \leq r < \dim_R(M)$, $\mathcal{L}^r(M)$ is non-empty by Corollary 4.2. In the following remark, it is shown that the set $\mathcal{L}^r(M)$ is independent of the module structure.

Remark 4.5. *Assume that L, M and N are finite R–modules and that r is a non–negative integer. Then the following statements are true.*

(i) *If* $\operatorname{Supp}_R(N) \subseteq \operatorname{Supp}_R(M)$, *then* $\mathcal{L}^r(N) \subseteq \mathcal{L}^r(M)$.

(ii) *If* $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ *is an exact sequence, then* $\mathcal{L}^r(M) = \mathcal{L}^r(L) \cup \mathcal{L}^r(N)$.

(iii) $\mathcal{L}^r(M) = \bigcup\limits_{\mathfrak{p} \in \operatorname{Ass}_R(M)} \mathcal{L}^r(R/\mathfrak{p})$.

Proof. (i) Assume that $\mathfrak{a}$ is an ideal of R which is not in $\mathcal{L}^r(M)$; so that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is artinian for all $i \geq r$. Therefore $\mathrm{H}^i_{\mathfrak{a}}(N)$ is artinian for all $i \geq r$ by [1, Theorem 3.1], that is $\mathfrak{a}$ does not belong to $\mathcal{L}^r(N)$. Thus $\mathcal{L}^r(N) \subseteq \mathcal{L}^r(M)$ as desired.

(ii) By (i), $\mathcal{L}^r(M) \supseteq \mathcal{L}^r(L) \cup \mathcal{L}^r(N)$. Assume that $\mathfrak{a} \in \mathcal{L}^r(M)$. There exists an integer i, $i \geq r$, such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is not artinian. Now, by the exact sequence $\mathrm{H}^i_{\mathfrak{a}}(L) \longrightarrow \mathrm{H}^i_{\mathfrak{a}}(M) \longrightarrow \mathrm{H}^i_{\mathfrak{a}}(N)$, the other inclusion follows.

(iii) By (i), we have the inclusion $\mathcal{L}^r(M) \supseteq \cup_{\mathfrak{p} \in \operatorname{Ass}_R(M)} \mathcal{L}^r(R/\mathfrak{p})$. Assume, conversely, that $\mathfrak{b} \notin \cup_{\mathfrak{p} \in \operatorname{Ass}_R(M)} \mathcal{L}^r(R/\mathfrak{p})$. There is a prime filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_s = M$ of M such that, for all $j \in \{1, \cdots, s\}$, $M_j/M_{j-1} \cong R/\mathfrak{p}_j$ for some $\mathfrak{p}_j \in \operatorname{Supp}_R(M)$. For each $j \in \{1, \cdots, s\}$, there is $\mathfrak{q}_j \in \operatorname{Ass}_R(M)$ contained in $\mathfrak{p}_j$ and thus, by assumption and part (i), $\mathfrak{b} \notin \mathcal{L}^r(R/\mathfrak{p}_j)$. Now, by applying $\mathrm{H}^i_{\mathfrak{b}}(-)$ on each exact sequence

$$0 \longrightarrow M_j \longrightarrow M_{j+1} \longrightarrow M_{j+1}/M_j \longrightarrow 0,$$

it follows that $\mathfrak{b} \notin \mathcal{L}^r(M)$. □

Before bringing the main theorem of this section, recall the following result which is straightforward from the fact that, for an R–module X and for each $\alpha \in R$, the kernel (respectively, the cokernel) of the natural map $X \longrightarrow X_\alpha$ is $\mathrm{H}^0_{R\alpha}(X)$ (respectively, $\mathrm{H}^1_{R\alpha}(X)$), where X_α denote the localization of X at set $\{1, \alpha, \alpha^2, \alpha^3, \cdots\}$.

Proposition 4.6. *For any R–module X and for any $\alpha \in R$, there are exact sequences*

$$0 \longrightarrow \mathrm{H}^1_{R\alpha}(\mathrm{H}^{i-1}_{\mathfrak{a}}(X)) \longrightarrow \mathrm{H}^i_{\mathfrak{a}+R\alpha}(X) \longrightarrow \mathrm{H}^0_{R\alpha}(\mathrm{H}^i_{\mathfrak{a}}(X)) \longrightarrow 0,$$

for all $i \geq 0$.

Proof. See [4, Proposition 8.1.2] (see also [3, Theorem 2.5]). □

The ith Bass number of X with respect to the prime ideal $\mathfrak{p}$ of R, denoted by $\mu^i(\mathfrak{p}, X)$, is defined to be the number of copies of the indecomposable injective module $\mathrm{E}_R(R/\mathfrak{p})$ in the direct sum decomposition of the ith term of a minimal injective resolution of X, which is equal to the rank of the vector space $\operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), X_\mathfrak{p})$ over the field $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. When $(R, \mathfrak{m})$ is local, we write $\mu^i(X) := \mu^i(\mathfrak{m}, X)$ and refer it the ith Bass number of X.

In the following theorem, we study Bass numbers of certain non–artinian local cohomology modules.

Theorem 4.7. *Assume that $(R, \mathfrak{m})$ is a local ring and that M is a finite R–module with Krull dimension d. Let $r < d$ be a fixed non-negative integer. Then for each maximal element $\mathfrak{q}$ of the non-empty set $\mathcal{L}^r(M)$,*

(i) $\mu^j(\mathrm{H}^i_{\mathfrak{q}}(M)) < \infty$ *for all* $j \geq 0$ *and all* $i \geq r$.
(ii) $\mathfrak{q}$ *is a prime ideal.*

Proof. (i) As $\mathrm{H}^i_{\mathfrak{m}}(M)$ is artinian for all $i \geq 0$, we have $\mathfrak{q} \neq \mathfrak{m}$. Choose an element $x \in \mathfrak{m} \setminus \mathfrak{q}$. Thus $\mathrm{H}^i_{\mathfrak{q}+Rx}(M)$ is artinian for all $i \geq r$. Using the exact sequence

$$0 \longrightarrow \mathrm{H}^1_{Rx}(\mathrm{H}^{i-1}_{\mathfrak{q}}(M)) \longrightarrow \mathrm{H}^i_{\mathfrak{q}+Rx}(M) \longrightarrow \mathrm{H}^0_{Rx}(\mathrm{H}^i_{\mathfrak{q}}(M)) \longrightarrow 0,$$

it follows that, for each $i \geq r$, the modules $\mathrm{H}^1_{Rx}(\mathrm{H}^i_{\mathfrak{q}}(M))$ and $\mathrm{H}^0_{Rx}(\mathrm{H}^i_{\mathfrak{q}}(M))$ are artinian and so they have finite Bass numbers. It follows by [8, Theorem 2.1] that $\mu^j(\mathrm{H}^i_{\mathfrak{q}}(M)) < \infty$ for all $j \geq 0$ and all $i \geq r$.

(ii) Assume that $x, y \in \mathfrak{m} \setminus \mathfrak{q}$ such that $xy \in \mathfrak{q}$. As $\mathfrak{q} + Rx$ and $\mathfrak{q} + Ry$ properly contain $\mathfrak{q}$, it follows that the modules $\mathrm{H}^i_{\mathfrak{q}+Rx}(M)$, $\mathrm{H}^i_{\mathfrak{q}+Ry}(M)$, and $\mathrm{H}^i_{\mathfrak{q}+Rx+Ry}(M)$ are artinian for all $i \geq r$. Applying the Mayer-Vietoris exact sequence

$$\mathrm{H}^i_{\mathfrak{q}+Rx}(M) \oplus \mathrm{H}^i_{\mathfrak{q}+Ry}(M) \longrightarrow \mathrm{H}^i_{(\mathfrak{q}+Rx)\cap(\mathfrak{q}+Ry)}(M) \longrightarrow \mathrm{H}^{i+1}_{\mathfrak{q}+Rx+Ry}(M),$$

we find that $\mathrm{H}^i_{(\mathfrak{q}+Rx)\cap(\mathfrak{q}+Ry)}(M)$ is artinian for $i \geq r$. Note that

$$\begin{aligned} \sqrt{\mathfrak{q}} &\subseteq \sqrt{(\mathfrak{q}+Rx)\cap(\mathfrak{q}+Ry)} \\ &= \sqrt{(\mathfrak{q}+Rx)(\mathfrak{q}+Ry)} \\ &= \sqrt{\mathfrak{q}^2 + \mathfrak{q}x + \mathfrak{q}y + Rxy} \\ &\subseteq \sqrt{\mathfrak{q}}. \end{aligned}$$

and hence $\mathrm{H}^i_{(\mathfrak{q}+Rx)\cap(\mathfrak{q}+Ry)}(M) \cong \mathrm{H}^i_{\mathfrak{q}}(M)$ is artinian for $i \geq r$. This contradicts the fact that $\mathfrak{q} \in \mathcal{L}^r(M)$, and so $\mathfrak{q}$ is a prime ideal. □

There have been many attempts in the literature made to find some conditions for the ideal $\mathfrak{a}$ to have finiteness for the Bass numbers of the local cohomology modules supported at $\mathfrak{a}$. In [5, Corollary 2], Delfino and Marley showed that the Bass number $\mu^i(\mathfrak{p}, \mathrm{H}^j_{\mathfrak{a}}(M))$ is finite for all $\mathfrak{p} \in \operatorname{Spec} R$ and all i, j whenever M is a finite module over a ring R and $\mathfrak{a}$ is an ideal of R with $\dim R/\mathfrak{a} = 1$.

Assume that $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of a local ring $(R, \mathfrak{m})$ with $\dim(R/\mathfrak{a}) = \dim(R/\mathfrak{b}) = 1$ such that $\mathrm{V}(\mathfrak{a}+\mathfrak{b}) = \{\mathfrak{m}\}$. Write the Mayer-Vietoris exact sequence

$$\mathrm{H}^j_{\mathfrak{m}}(M) \longrightarrow \mathrm{H}^j_{\mathfrak{a}}(M) \oplus \mathrm{H}^j_{\mathfrak{b}}(M) \longrightarrow \mathrm{H}^j_{\mathfrak{a}\cap\mathfrak{b}}(M) \longrightarrow \mathrm{H}^{j+1}_{\mathfrak{m}}(M).$$

As $\mathrm{H}^i_{\mathfrak{m}}(M)$ is artinian for all i, we find that $\mathrm{H}^j_{\mathfrak{a}\cap\mathfrak{b}}(M)$ has finite Bass numbers if and only if both $\mathrm{H}^j_{\mathfrak{a}}(M)$ and $\mathrm{H}^j_{\mathfrak{b}}(M)$ have finite Bass numbers. Therefore [5, Corollary 2] is equivalent to the case where the ideal $\mathfrak{a}$ is prime.

Comment. Assume that $\mathfrak{p}$ is a prime ideal of R such that $\dim(R/\mathfrak{p}) = 1$ and r is the smallest integer (if there is any) such that $\mathrm{H}^i_{\mathfrak{p}}(M)$ is not artinian. Thus $\mathfrak{p}$ is a maximal element of $\mathcal{L}^r(M)$. By Theorem 4.7, $\mu^j(H^i_{\mathfrak{p}}(M)) < \infty$ for all $j \geq 0$ and all $i \geq r$. As $\mathrm{H}^i_{\mathfrak{p}}(M)$ is artinian for all $i < r$, all $\mathrm{H}^i_{\mathfrak{p}}(M)$ have finite Bass numbers. Thus Theorem 4.7 generalizes [5, Corollary 2].

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